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Brans–Dicke Cosmological Exact Solution in a Radiation-Filled Robertson–Walker Universe

R. T. Singh[1] and Shridhar Deo[1]

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Considering a Robertson–Walker line element, exact solutions are obtained for radiation-filled cosmological differential equations of Brans–Dicke theory with the assumption that the radius of curvature Q of the universe varies directly as the nth power of time. The solution is found to be valid for closed space only and the coupling constant w of the scalar tensor theory is necessarily negative. The radius of curvature of increases linearly with respect to . ᵓ age of the universe, while the gravitational constant k varies directly as the s uare of the radius of the universe. The solution obtained is in contradiction to Dirac's hypothesis, in which the gravitational constant should decrease with time in an expanding universe.

1. INTRODUCTION

Studies of the cosmological solutions of Brans–Dicke (BD) scalar tensor theory with various relations between the gravitational constant k and the radius of curvature of the Universe have been made by various workers. Morganstern (1971a–c) has analyzed the consequences of unit transformation in view of the new BD field equations from the viewpoint of the cosmological interpretation in an expanding Friedmann universe governed by the equation of state $P = \varepsilon\rho$ $(0 \leq \varepsilon \leq 1)$ by assuming a power law solution. Dehnen and Obregon (1971, 1972a) have discussed the cosmological solutions of the BD theory for the case where a power law is valid between the gravitational constant G and the radius of curvature Q of the universe under the assumption that the present mass density of the universe does not vanish. In another paper, Dehnen and Obregon (1972b) have further shown that for all values of $\omega > -3/2$, there exist cosmological vacuum solutions of closed three-dimension space of positive curvature.

[1]Department of Mathematics, Manipur University, Canchipur, Imphal-795003 (Manipur), India.

In the present paper, we show that exact solutions can be obtained for the cosmological equations of Brans–Dicke scalar field in the presence of a disordered distribution of radiation with the assumpton that for an expanding universe the radius of the universe varies directly as the nth power of the age of the universe. The solutions obtained turn out to be the same as obtained by Obregon and Chaevet (1978) with the assumption of a power law ($kQ^n = \text{const}$). The radius of the universe increases linearly with respect to the age of the universe and the gravitational constant varies quadratically with time. The BD scalar field ϕ is found to vary inversely as the square of the age of the universe as t tends to infinity and ϕ tends to zero. The solution obtained for the gravitational constant k contradicts Dirac's hypothesis that the gravitational constant k should be inversely proportional to the age of the universe.

2. COSMOLOGICAL FIELD EQUATIONS

Let us consider the Robertson–Walker line element

$$ds^2 = dt^2 - Q^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2\right) \tag{1}$$

with $K = +1, 0, -1$, for the space of positive, vanishing, and negative curvature index, respectively. The Einstein field equations of the Brans–Dicke theory for a radiation-filled universe are given by

$$\left(\frac{\dot Q}{Q} - \frac{\dot k}{k}\right)^2 + \frac{K}{Q^2} = -2\frac{\ddot Q}{Q} + \frac{\ddot k}{k} + \frac{\dot k}{k}\left[-\left(1+\frac{\omega}{2}\right)\frac{\dot k}{k} + \frac{E}{Q}\right] \tag{2}$$

$$\left(\frac{\dot Q}{Q} - \frac{1}{2}\frac{\dot k}{k}\right)^2 + \frac{K}{Q^2} = \frac{\dot k}{k}\left[\frac{1}{4}\left(1+\frac{2\omega}{3}\right)\frac{\dot k}{k} - \frac{E}{Q}\right] \tag{3}$$

$$\frac{\dot k}{k}Q^3 = -B, \qquad B = \text{const} \tag{4}$$

and

$$E = \left(\frac{3+2\omega}{4+2\omega}\frac{c^2}{3}\right)\frac{F}{B}, \qquad F = \rho Q^4 = \text{const} \tag{5}$$

Equations (2)–(4) differ from the original Brans–Dicke equations, since in place of the scalar variable ϕ we use

$$k = \frac{8\pi}{c^4}\phi^{-1}\left(\frac{4+2\omega}{3+2\omega}\right) \tag{6}$$

3. SOLUTION OF THE FIELD EQUATIONS AND THEIR PHYSICAL INTERPRETATIONS

To solve the three equations (2)-(4) we assume that

$$Q = dt^n \tag{7}$$

where d is a positive constant of variation and n is a positive real number. Using (7) in (4), we obtain

$$k = \frac{d^3(1-3n)}{8} t^{3n-1} \tag{8}$$

where the arbitrary constant of integration is taken to be zero. Using (7) and (8) in (2) and (3), we obtain the equations

$$\frac{(1-2n)^2}{t^2} + \frac{K}{d^2 t^{2n}} = \frac{-2n(n-1)}{t^2} + \frac{(3n-1)(3n-2)}{t^2} + \frac{3n-1}{t}\left[-\left(1+\frac{\omega}{2}\right)\left(\frac{3n-1}{t}\right) + \frac{E}{dt^n}\right] \tag{9}$$

$$\frac{(1-n)^2}{4t^2} + \frac{K}{d^2 t^{2n}} = \frac{3n-1}{t}\left[\frac{1}{4}\left(1+\frac{2\omega}{3}\right)\left(\frac{3n-1}{t}\right) - \frac{E}{dt^n}\right] \tag{10}$$

If we are to obtain relations between the constants only from (9) and (10), the only possible value of n is 1.

Taking $n = 1$ in (9) and (10), we derive the relations

$$(3+2\omega)d^2 - 2Ed + K = 0 \tag{11}$$

$$\left(\frac{3+2\omega}{-3}\right) d^2 + 2Ed + K = 0 \tag{12}$$

respectively.

Case I

Taking $K = 1$ in (11) and (12), we obtain the solution

$$d = \left(\frac{-3}{3+2\omega}\right)^{1/2} \tag{13}$$

$$E = -\left(-\frac{3+2\omega}{3}\right)^{1/2} \tag{14}$$

respectively.

Using (13) and taking $n = 1$ in (7), we obtain

$$Q = \left(\frac{-3}{3+2\omega}\right)^{1/2} t \tag{15}$$

Using (14) and (15) and taking $n = 1$ in (8), we obtain

$$k = -18 \frac{4+2\omega}{(3+2\omega)^2} \frac{t^2}{c^2 F} \tag{16}$$

The corresponding solutions for ρ, P, and ϕ are given by

$$\rho = F\left(-\frac{3+2\omega}{3}\right)^4 \frac{1}{t^4} \tag{17}$$

$$P = \frac{F}{3}\left(-\frac{3+2\omega}{3}\right)^4 \frac{1}{t^4} \tag{18}$$

$$\phi = -\frac{4\pi}{gc^2}(3+2\omega)\frac{F}{t^2} \tag{19}$$

respectively.

Case II

Taking $K = 0$ in (11) and (12), we obtain

$$\omega = -3/2, \qquad E = 0$$

The corresponding solutions for Q, k, ρ, P, and ϕ are given by

$$Q = dt \tag{20}$$

$$k = -2\frac{d^3}{B} t^2 \tag{21}$$

$$\rho = \frac{F}{d^4 t^4} \tag{22}$$

$$P = \frac{1}{3}\frac{F}{d^4 t^4} \tag{23}$$

$$\phi = -\frac{4\pi}{c^4}\left(\frac{4+2\omega}{3+2\omega}\right)\frac{B}{d^3 t^2} = \infty \tag{24}$$

since $3+2\omega = 0$, and where B is taken to be negative.

Case III

Taking $K = -1$, we obtain from (11) and (12) the solutions

$$d = \left(\frac{3}{3+2\omega}\right)^{1/2} \tag{25}$$

$$E = \left(\frac{3+2\omega}{3}\right)^{1/2} \tag{26}$$

Then the corresponding solutions for Q, k, ρ, P, and ϕ are given by

$$Q=\left(\frac{3}{3+2\omega}\right)^{1/2} t \tag{27}$$

$$k=\frac{-18(4+2\omega)t^2}{(3+2\omega)^2c^2F} \tag{28}$$

$$\rho=\frac{F(3+2\omega)^2}{gt^4} \tag{29}$$

$$P=\frac{F(3+2\omega)^2}{27t^4} \tag{30}$$

$$\phi=-\frac{4\pi}{gc^2}\frac{F}{t^2}(3+2\omega) \tag{31}$$

Therefore

$$kQ^{-2}=\frac{-6(4+2\omega)}{(3+2\omega)c^2F}=\text{const} \tag{32}$$

In case I, from (15) we find that for real solutions ω will be necessarily negative.

When $K=1$, for real solutions

$$-\infty<\omega<-3/2$$

We also further observe from (15) that the radius of the universe increases linearly with the age of the universe. From (16) we see that the gravitatinal constant k varies directly as the square of the age of the universe, which is in contrast to Dirac's hyphothesis. As $t\to\infty$, ρ, P, and ϕ tend to zero and the universe becomes infinitely large. The power law derived from the solution is that

$$kQ^{-2}=\frac{6(4+2\omega)}{(3+2\omega)c^2F} \tag{33}$$

which is constant.

In case II, we see that the solution is valid for the coupling constant $\omega=-3/2$. Now we observe that for a cosmological flat universe, the radius of the universe, the gravitational constant, the pressure, and the density are all independent of the coupling constant ω. Taking the constant B to be negative, we see that the BD scalar ϕ will remain infinite while the gravitational constant k increases quadratically with respect to the age of the universe.

The power law derived from the solution is

$$kQ^{-2}=-2d/B=\text{const} \tag{34}$$

In case III, we find that corresponding to negative value of the curvature index K, the radius of the universe is found to increase linearly with the age of the universe and the gravitational constant k varies directly as the square of the age of the universe. For real solutions the value of the coupling constant ω must be greater than $-3/2$.

Since the mass density ρ is to be always positive, the constant F must be positive.

From equation (28) we observe that the gravitational constant k must always be negative, since $3+2\omega>0$ and $F>0$.

So, from (31) we find that the BD scalar ϕ is always negative. Here ρ, P, and ϕ all tend to zero as $t\to\infty$. The power law for this case is

$$kQ^{-2}=\frac{-6(4+2\omega)}{(3+2\omega)c^2F}=\text{const} \tag{35}$$

In all three cases we can draw the conclusion that the gravitational variable varies inversely as the square of the radius of the universe.

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